Between Maharam's and von Neumann's problems*

Ilijas Farah [†]   Jindřich Zapletal [‡]
York University   University of Florida

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Abstract

We show that in the definable context the Maharam and von Neumann problems essentially coincide. We also prove that the random forcing is the only definable ccc forcing adding a single real that does not make the ground model reals null, and that the only pairs of definable ccc σ-ideals with the Fubini property are $(\texttt{meager}, \texttt{meager})$ and $(\texttt{null}, \texttt{null})$.

In Scottish Book, von Neumann asked whether every ccc, weakly distributive complete Boolean algebra carries a strictly positive probability measure. In her commentary to von Neumann's problem ([16]) D. Maharam pointed out that this problem naturally splits into two problems: (a) whether all such algebras carry a strictly positive continuous submeasure, and (b) whether every algebra that carries a strictly positive continuous submeasure carries a strictly positive measure. The latter problem is known under the names of Maharam's Problem and Control Measure Problem (see [8], [4, §393]). While von Neumann's problem has a consistently negative answer ([15]), Maharam's problem can be stated as a $\mathbf{\Sigma}_2^1$ statement and is therefore by Shoenfield's theorem absolute between transitive models of set theory containing all countable ordinals.

Very recently Balcar, Jech and Pazák announced ([1]) that under the P-ideal dichotomy ([23]) every c.c.c. weakly distributive complete Boolean algebra carries a strictly positive Maharam submeasure. About a month before their announcement, we discovered a similar result for suitably definable forcings (see below for definitions).

Theorem 0.1. *Let I be a c.c.c. σ-ideal on Borel subsets of 2^ω that is analytic on G_δ. The following are equivalent:*

- *P_I is a weakly distributive notion of forcing*

- *there is a Maharam submeasure on 2^ω such that I is the σ-ideal of its null sets.*

A suitable large cardinal assumption implies that the assumption that I is analytic on G_δ can be relaxed to 'I is definable.'

By a result of Shelah ([20]), the first clause is equivalent to saying that P_I does not add Cohen reals, so we have a dichotomy for definable ccc forcing notions of the form P_I. Here P_I is the partial ordering of I-positive Borel sets under inclusion. Thus, the von Neumann's problem restricted to definable partial orders of the form P_I coincides with the Control Measure Problem. By an absoluteness argument, this also follows from Balcar–Jech–Pazák's result.

We were able to draw an interesting consequence of the theorems. In order to state it succinctly we quote the large cardinal version.

Corollary 0.2 (LC). *Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset \mathbb{R} \times \mathbb{R}$ with all vertical sections in I and all horizontal sections of full Lebesgue measure.*

2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the random forcing.*

In other words, if P_I does not force that the set of ground model reals is null, then P_I is the random forcing. Modulo Theorem 0.1, this is really a consequence of a result of Christensen ([3]). There is a curious duality with an earlier result of Shelah that a similar result holds on the meager side.

Fact 0.3 (LC). *([19], see also [25]) Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset \mathbb{R} \times \mathbb{R}$ with all vertical sections in I and all horizontal sections comeager.*

2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the Cohen forcing.*

Another attractive corollary is that the only definable c.c.c. σ-ideals for which Fubini theorem holds are the meager and null ideal (Theorem 3.3). This shows that the only 'reasonable' ideals as introduced by Kunen in [13] are `meager` and `null`.

Terminology

The notation in the paper follows the set theoretic standard of [7]. An ideal I is analytic on G_δ if for every G_δ set $A \subseteq 2^\omega \times 2^\omega$ the set of all x such that the vertical section of A at x is in I is analytic. Note that both `meager` and `null` are analytic on G_δ (see [11]). Throughout the paper we will say that an ideal is *definable* if it belongs to $L(\mathbb{R})$. The suitable large cardinal assumption

in Theorem 0.1 if I is in $L(\mathbb{R})$ is that there are ω Woodin cardinals with a measurable above them all. In all the subsequent results of this note no large cardinal assumptions are needed if I is assumed to be analytic on G_δ. For large cardinals and $L(\mathbb{R})$ see e.g., [10].

A *Maharam submeasure* (or a *continuous submeasure*) on a complete Boolean algebra $\mathcal{B}$ is a function ϕ such that

1. $A \subseteq B$ implies $\phi(A) \le \phi(B)$,

2. $\phi(A \cup B) \le \phi(A) + \phi(B)$,

3. $\phi(0_{\mathcal{B}}) = 0$, and

4. if A_n is a decreasing sequence in $\mathcal{B}$ then $\phi(\bigcap_n A_n) = \lim_n \phi(A_n)$.

An algebra that carries a strictly positive Maharam submeasure is called a *submeasure algebra*.

A forcing notion is *bounding* (or *weakly distributive*) if every element of ω^ω in the extension is dominated by a ground-model function in ω^ω. We use the words "bounding" and "weakly distributive" interchangeably. It is entirely irrelevant which uncountable Polish space the ideals in question measure; our choice is the Cantor space 2^ω for definiteness and ease of notation. To weed out trivial cases, we assume that ideals contain all singletons.

1   The proof of Theorem 0.1

It is a standard fact that if I is the ideal of null sets of some Maharam submeasure, then the poset P_I is weakly distributive (see e.g., [4, 392I]). Suppose now that I is a definable, weakly distributive c.c.c. σ-ideal on Borel subsets of 2^ω. To find a Maharam submeasure generating the σ-ideal I we will use two ingredients. One is almost trivial:

Fact 1.1 ([26] Lemma 2.2.3.). *Suppose that I is a σ-ideal on 2^ω such that P_I is proper. The following are equivalent:*

- *P_I is weakly distributive*

- *compact sets are dense (every I-positive Borel set has an I-positive compact subset) and continuous reading of names (for every I-positive Borel set B and a Borel function $f : B \to \omega^\omega$ there is an I-positive set $C \subset B$ such that $f \restriction C$ is continuous).*

Note that this implies that our ideal I has a basis consisting of G_δ sets. For let $A \in I$ be a Borel set. The collection of compact I-positive sets disjoint from A is dense in P_I: for every I-positive Borel set B, the set $B \setminus A$ is still Borel and I-positive and therefore it has a compact I-positive subset. Choose then a maximal antichain X consisting of such compact sets. Since the poset P_I

is c.c.c., it is the case that the antichain X is countable, and $2^\omega \setminus \bigcup X$ is the required G_δ set in the ideal I covering the set A.

The other ingredient is a result of Solecki. For an ideal I on 2^ω let $\hat{I}$ be the collection of subsets of $2^{<\omega}$ defined by putting $a \in \hat{I}$ if the set $B_a = \{r \in 2^\omega : \text{for infinitely many } n, r \restriction n \in a\}$ is in I. It is immediate that $\hat{I}$ is an ideal, because $B_{a \cup b} = B_a \cup B_b$ and so if both $B_a, B_b \subset 2^\omega$ are in the σ-ideal I, so is $B_{a \cup b}$.

Fact 1.2. *Suppose that I is a σ-ideal on Borel subsets of 2^ω that is analytic on G_δ. The following are equivalent:*

- *$\hat{I}$ is a P-ideal and I has a basis consisting of G_δ sets*

- *There is a Maharam submeasure on 2^ω such that I is the collection of its null sets.*

Furthermore, large cardinals imply that this equivalence holds for every definable ideal I.

Proof. This was proved in [21, Theorem 5.2]. in the case when I is analytic on G_δ. The definability assumption was used in this proof only to show that $\hat{I}$ is analytic, and assuming large cardinals, in [22, Theorem 4] it was proved that all definable P-ideals are analytic. $\qquad\square$

Fact 1.2 clearly implies that we will be done once we prove that $\hat{I}$ is a P-ideal. To verify that this holds, fix a collection $\{a_n : n \in \omega\} \subset \hat{I}$ and aim to construct some set $b \in \hat{I}$ which includes each of them up to a finite set.

Claim 1.3. *The collection of compact I-positive sets C such that their associated tree on $2^{<\omega}$ has a finite intersection with each set $a_n : n \in \omega$, is dense in P_I.*

Proof. Suppose $A \in P_I$ is a positive Borel set, and let $B = A \setminus \bigcup_n B_{a_n}$. B is still an I-positive Borel set, and the function $f : B \to \omega^\omega$, $f(r)(n) = \max\{m \in \omega : r \restriction m \in a_n\}$, is Borel and well-defined on it. By Fact 1.1, there is an I-positive compact set $C \subset B$ such that $f \restriction C$ is continuous. By a compactness argument, for every number n the set $\{f(r)(n) : r \in C\}$ is finite. The claim follows. $\qquad\square$

Let X be a maximal antichain of I-positive compact sets from the claim. Since the poset P_I is c.c.c., the collection X is countable, enumerated as $\{C_k : k \in \omega\}$. Let $T_k \subset 2^{<\omega}$ be the tree associated with the compact set C_k for every $k \in \omega$. Finally, let $b \subset 2^{<\omega}$ be the set $\bigcup_n (a_n \setminus \bigcup_{k \in n} T_k)$. It is clear that the set b includes every a_n up to a finite piece. To show that $B_b \in I$ and $b \in \hat{I}$, note that for every number $k \in \omega$ the intersection $T_k \cap b = \bigcup_{n \leq k} a_n$ is finite, and so the set B_b is disjoint from $\bigcup X$. However, the antichain $X \subset P_I$ was chosen to be maximal, and therefore the set $2^\omega \setminus \bigcup X$ is I-small and so is its subset B_b. The theorem follows.

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2 Fubini failing

A submeasure ϕ is *pathological* if it does not dominate a positive nonzero finitely additive functional. A *control measure* for a Maharam submeasure ϕ is a measure μ that has the same null sets as ϕ. A Maharam submeasure is *Borel* if it is defined on the Borel algebra on 2^ω. A submeasure is *diffuse* if all countable sets are null. All results of this section are probably well-known.

Lemma 2.1. *The following are equivalent for a diffuse Borel Maharam submeasure ϕ.*

1. ϕ *is pathological.*

2. *There is a ϕ-positive set B such that the restriction of ϕ to B has a control measure.*

3. *There is a ϕ-positive set B such that $P_{\mathrm{Null}(\phi)}$ is forcing equivalent to random below B.*

Proof. Let us write $I = \mathrm{Null}(\phi)$. Assume (1), so there is a nonzero finitely additive functional $\nu \le \phi$ dominated by ϕ. There are two cases.

Assume there is a ϕ-positive set B such that $\nu(C) \ne 0$ for every I-positive set $C \subset B$. Then Borel/I is weakly distributive (see e.g., [4, 392I]). By [4, 391D] there is a strictly positive measure on Borel/I, and therefore (2) holds.

Otherwise, every ϕ-positive set B contains a ϕ-positive set C such that $\nu(C) = 0$. In this case, choose a maximal antichain $\{C_n : n \in \omega\}$ of sets such that $\phi(C_n) > 0$ and $\nu(C_n) = 0$, enumerated using the ccc of Borel/I. Consider the sets $D_m = \bigcup_{n > m} C_n$. By the finite additivity of the functional ν it is the case that $\nu(D_m) = \nu(2^\omega)$ for all $m \in \omega$. By the continuity of the submeasure ϕ, the numbers $\phi(D_m)$ converge to zero, since the D_m's for a decreasing collection of sets with empty intersection. This however contradicts the assumption $\nu \le \phi$.

Clause (2) implies (1) by [5, Theorem 2]. The equivalence of (2) and (3) follows by Maharam's theorem. $\square$

Lemma 2.2. *If ϕ is a Borel Maharam submeasure then there is a Borel set A such that ϕ has a control measure on B and is pathological on $B^{\complement}$.*

Proof. Find a maximal family $\mathcal{F}$ of pairwise orthogonal measures dominated by ϕ, and let B be the union of their supports. By the ccc-ness of Borel/$\mathrm{Null}(\phi)$, $\mathcal{F}$ is countable. If $\mathcal{F} = \{\mu_i | i \in \omega\}$ then $\sum_i 2^{-i} \mu_i$ is a control measure for ϕ on B. By Lemma 2.1, ϕ is pathological on the complement of B. $\square$

Lemma 2.3 below was roughly proved by Christensen [3, Theorem 6]. We shall use his result. Let μ denote the Lebesgue measure on $[0, 1]$; the choice is immaterial as any other diffuse Borel probability measure would do.

Lemma 2.3. *Suppose I is the null ideal for some Maharam Borel submeasure ϕ on 2^ω. Exactly one of the following holds:*

1. *There is a ϕ-positive Borel set B such that the restriction of ϕ to B has a control measure.*

2. *There is a Borel set $C \subseteq [0,1] \times 2^\omega$ such that $\phi(C_x) = 0$ for all $x \in [0,1]$ and $\mu([0,1] \setminus C^y) = 0$ for every $y \in 2^\omega$.*

Proof. By Fubini's theorem, (1) excludes (2). Suppose now that (1) fails. By Lemma 2.1, ϕ is pathological. Christensen proved in [3, Theorem 6], Theorem 6 that if ϕ is pathological then (2) holds. □

A submeasure ϕ on 2^ω is *normalized* if $\phi(2^\omega) = 1$.

Lemma 2.4. *Assume ψ is a normalized pathological Borel submeasure. Then for every $n \in \mathbb{N}$ there are pairwise disjoint sets A_i ($i < n$) of submeasure at least $1/3$ each.*

Proof. This was proved by Kalton and Roberts ([9]) for an unspecified $\varepsilon > 0$ in place of $1/3$, and sharpened by Louveau ([14]) to the present form. □

Lemma 2.5. *Assume ϕ and ψ are normalized diffuse Borel Maharam submeasures on 2^ω and ψ is pathological. Then there is a Borel set $C \subseteq 2^\omega \times 2^\omega$ such that $\psi(C_x) \geq 1/3$ for all $x \in 2^\omega$ and $\phi(C^y) = 0$ for all $y \in 2^\omega$.*

Proof. Since ϕ is diffuse and Maharam, every set of submeasure δ has a subset of submeasure ϵ for every $\epsilon \in [0, \delta]$. For each n a maximal antichain of Borel sets such that the submeasure of each one is between 2^{-n-1} and 2^{-n}. Since ϕ is Maharam, this antichain is finite and we can enumerate it as B_i^n ($i < k_n$). Using Lemma 2.4, fix a partition of 2^ω into Borel sets A_i^n ($i < k_n$) such that $\psi(A_i^n) \geq 1/3$ for all n and i. Let

$$C(n) = \bigcup_{i=0}^{k_n-1} B_i^n \times A_i^n$$

$$C = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} C(n).$$

Note that $\psi(C(n)_x) \geq 1/3$ and that $\phi(C(n)^y) \leq 2^{-n}$ for all x, y in 2^ω. Therefore for all x, y we have $\psi(C_x) \geq 1/3$ and $\phi(C^y) \leq \sum_{n=m}^{\infty} 2^{-n} = 2^{-m+1}$ for all m, hence $\phi(C^y) = 0$. □

Lemma 2.6. *Assume ϕ and ψ are diffuse Borel Maharam submeasures and ϕ does not have a control measure. Then there is a Borel set $A \subseteq 2^\omega \times 2^\omega$ such that $\psi(A_x) = 0$ for all x and $\inf_y \phi(A^y) > 0$.*

Proof. Let A be a Borel set such that the restriction of ϕ to $D^\complement$ has a control measure while the restriction of ϕ to D is pathological, as given by Lemma 2.2. By our assumption, $\phi(D) > 0$. Again using Lemma 2.2, find a Borel partition $2^\omega = B \cup C$ so that ψ has a control measure on B and is pathological on C. By Lemma 2.3 there is Borel $E \subseteq D \times B$ such that $\phi(E^y) = \phi(D)$ for all $y \in B$

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and $\psi(E_x) = 0$ for all x. By Lemma 2.5 there is a Borel $F \subseteq D \times C$ such that $\phi(F^y) \geq \frac{1}{3}\phi(D)$ for all $y \in C$ and $\psi(F_x) = 0$ for all x. Then $A = E \cup F$ is as required. $\qquad\square$

3 Non-commutativity

Given σ-ideals I and J on the real line, let $I \perp J$ be the statement that there is a Borel subset B of the plane such that all of its vertical sections are in the ideal J and all of the horizontal sections of the complement are in the ideal I. Thus $I \perp J$ means that the Fubini theorem between I and J fails in a particularly violent manner. If the σ-ideals I and J are definable and c.c.c., $I \perp J$ is easily seen to be equivalent to both $P_I \Vdash 2^\omega \cap V \in \dot{J}$ and $P_J \Vdash 2^\omega \cap V \in \dot{I}$ [26, 5.4.8].

Results of this section do not any require large cardinals if the ideals are assumed to be analytic on G_δ. This is because in this case both the compatibility and the incompatibility relations of P_I are analytic, and therefore the result of [20] applies. Let us recall and prove Corollary 0.2.

Corollary 3.1 (LC). *Suppose I is a definable c.c.c. σ-ideal on 2^ω. Then exactly one of the following holds:*

1. *There is a Borel set $B \subset \mathbb{R} \times \mathbb{R}$ with all vertical sections in I and all horizontal sections of full Lebesgue measure.*

2. *There is a condition $p \in P_I$ such that P_I below p is isomorphic to the random forcing.*

Proof. By Fubini's theorem, two clauses exclude each other. Assume that P_I is not isomorphic to the random algebra below any positive set B. By Theorem 0.1 and Lemma 2.3, we may assume P_I is not bounding, so by a result of Shelah ([20]) it adds a Cohen real. Let $f \colon 2^\omega \to 2^\omega$ be a Borel function such that the preimages of meager sets are in I. Fix a Borel set $B \subseteq 2^\omega \times 2^\omega$ whose all vertical sections are null and whose complements of horizontal sections are meager. Then the set $D = \{(x,y) | (f(x), y) \in A\}$ witnesses that $I \perp \mathtt{null}$. $\qquad\square$

We do not know whether $\mathrm{Null}(\phi) \perp \mathrm{Null}(\psi)$ whenever ϕ and ψ are Maharam submeasures at least one of which is pathological.

Shelah [19] defined the notion of commutation for definable c.c.c. σ-ideals I, J: they commute if for all reals r, s in all generic extensions of V, the statement "r is $V[s]$-generic for P_I and s is V-generic for P_J" is equivalent to "s is $V[r]$-generic for P_J and r is V-generic for P_I." (Note that in this situation r is automatically V-generic for P_I, since it avoids all sets in I coded in V.) Shelah proved that the only ideal commuting with $\mathtt{meager}$ is $\mathtt{meager}$ itself. Corollary 3.1 can be formulated by saying that the only ideal commuting with $\mathtt{null}$ is $\mathtt{null}$ itself. In [19], Problem 11.5, Shelah asked whether the only Suslin forcings that commute with themselves are cohen and random. (A forcing notion $\mathbb{P}$ is *suslin* if its underlying set is $\mathbb{R}$ and both $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are analytic subsets of the plane.

If I is analytic on G_δ, then P_I is easily Suslin.) By Theorem 3.3, the answer to this question restricted to definable forcings of the form P_I is positive.

Recław and Zakrzewski ([17]) say that a pair of ideals I, J has the *Fubini property* if for every Borel $B \subseteq 2^\omega \times 2^\omega$ such that $\{x | B_x \notin J\} = \emptyset$ we have $\{y | B^y \notin I\} \in J$. They have proved that in a certain restricted class of ccc σ-ideals of Borel sets (`meager`, `meager`) and (`null`, `null`) are the only pairs that have the Fubini property. They also found a consistent example (using a large cardinal assumption) of another ccc σ-ideal I such that (I, I) has the Fubini property, and asked whether there are other 'natural' examples of pairs of ccc ideals with Fubini property. Theorem 3.3 gives a negative answer to their question restricted to the class of definable ideals.

In order to give unified treatment of ideals `meager` and `null` and the corresponding forcing notions Cohen and random, in [13, Definition 1.26] Kunen introduced the class of 'reasonable' ideals. Among other properties, every reasonable ideal is a *Fubini ideal* ([13, Definition 1.3]) and this implies that (I, I) has the Fubini property. Therefore by Theorem 3.3, `meager` and `null` are the only reasonable ideals that are analytic on G_δ. The definition of reasonable also involves being absolute ([13, Definition 1.20]) and under large cardinals every absolute set of reals belongs to $L(\mathbb{R})$ by [6, Theorem 3.2]. Therefore large cardinals imply that `meager` and are are the only reasonable ideals.

By [18] if the assumption that I is a Fubini ideal is dropped from the definition of a reasonable ideal then there are many ideals satisfying the weaker notion.

Lemma 3.2. *Suppose I and J are definable c.c.c. σ-ideals on 2^ω. Then the following are equivalent.*

1. *P_I and P_J commute.*

2. *If $B \subseteq 2^\omega \times 2^\omega$ is Borel then $\{x | B_x \notin J\} \notin I$ implies $\{y | B^y \notin I\} \neq \emptyset$.*

3. *Pair J, I has the Fubini property.*

Proof. Assume (2) fails and fix a Borel B such that $\{x | B_x \notin J\} \notin I$ and $C = \{y | B^y \notin I\} \in J$. Let $A = B \setminus 2^\omega \times C$, and note that $\{x | A_x \notin J\} \notin I$ and $\{y | A^y \notin I\} = \emptyset$. Let x be V-generic for P_I so that that $A_x \notin J$ and let $y \in A_x$ be $V[x]$-generic for P_J. Since $A^y \in I$ and $x \in A^y$, x is not P_I-generic over $V[y]$.

Now assume (1) fails, and fix a countable transitive model M of a large enough fragment of ZFC containing definitions of I and J. Since $\{x | x$ is M-generic for $P_I\}$ is equal to the complement of the union of all Borel sets coded in M that belong to I, it is Borel. Similarly, the set

$$A_{IJ} = \{(x, y) | x \text{ is } M\text{-generic for } P_I \text{ and } y \text{ is } M[x]\text{-generic for } P_J\}$$

is Borel, and $B = A_{IJ} \setminus \{(x, y) | (y, x) \in A_{JI}\}$ is a Borel set consisting of all pairs (x, y) that fail the commutativity condition. This set is nonempty by our assumption, and it satisfies (2).

To see that (2) and (3) are equivalent, take the contrapositive of (2). $\qquad\square$

Theorem 3.3 (LC). *Suppose I, J are definable c.c.c. σ-ideals on 2^ω. Then one of the following holds:*

1. *Both P_I and P_J are isomorphic to the Cohen algebra.*

2. *Both P_I and P_J are isomorphic to the Lebesgue measure algebra.*

3. *P_I and P_J do not commute.*

In particular, if P_I of this kind commutes with itself, then it is either Cohen or random.

The proof of Theorem 3.3 breaks into several cases according to whether the posets P_I, P_J are bounding or not, with wildly different arguments in each case.

Lemma 3.4 (LC). *Suppose I, J are definable c.c.c. σ-ideals on 2^ω such that both forcings P_I and P_J add an unbounded real. Exactly one of the following holds:*

- *there are Borel I-positive set B and a Borel J-positive set C such that both P_I below B and P_J below C are isomorphic to the Cohen algebra*

- *$I \perp J$*

Proof. There is nothing really new here. Clearly the first item implies the failure of $I \perp J$. On the other hand, suppose that the first item fails. Then one of the partial orders, P_I say, is not isomorphic to the Cohen algebra below any condition. By [19] 9.16 or [25] 6.6, $P_I \Vdash 2^\omega \cap V$ is meager, so $I \perp \mathtt{meager}$ and there is a Borel set $E \subset 2^\omega \times 2^\omega$ such that its vertical sections are meager and the horizontal sections of its complement are I-small. By [20], 1.14, P_J adds a Cohen real over V and so there is a Borel function $f : 2^\omega \to 2^\omega$ such that preimages of meager sets are J-small. It is not difficult to verify that the Borel set $D \subset 2^\omega \times 2^\omega$ defined by $\langle x, y \rangle \in D$ if and only if $\langle x, f(y) \rangle \in E$ witnesses $I \perp J$. The lemma follows. $\square$

Lemma 3.5 (LC). *Suppose I, J are definable c.c.c. σ-ideals such that both forcings P_I and P_J are bounding. If P_I is not equivalent to random, then there is a Borel $B \subseteq 2^\omega \times 2^\omega$ such that $B_x \in J$ for all x and $B^y \notin I$ for all y.*

Proof. By Theorem 0.1, both I and J are null ideals for some Maharam submeasures ϕ and ψ, respectively. By Lemma 2.1, ϕ does not have a control measure. Therefore we are in the situation of Lemma 2.6. $\square$

Lemma 3.6 (LC). *Suppose that I, J are definable c.c.c. σ-ideals on 2^ω such that P_I is bounding while P_J adds an unbounded real. Then $I \perp J$.*

Proof. By Theorem 0.1, there is a Maharam submeasure ϕ such that I is the null ideal for ϕ. We will first prove that $I \perp \mathtt{meager}$. For $s \in 2^n$ let $[s] = \{x \in 2^\omega | x \upharpoonright n = s\}$.

Claim 3.7. *If ϕ is a Maharam submeasure on the Borel algebra of 2^ω, then for every $\varepsilon > 0$ there is $m_\varepsilon \in \mathbb{N}$ such that $\phi([s]) \leq \varepsilon$ for every $s \in 2^{m_\varepsilon}$.*

Proof. Assume not, and find $s_m \in 2^m$ such that $\phi([s_m]) \geq \varepsilon$ for all m. Ramsey's theorem gives us two possibilities.

Either there is an infinite set $B \subseteq \omega$ such that $[s_m]$ $(m \in B)$ are pairwise disjoint. In this case the open sets $U_n = \bigcup\{[s_m] | m \geq n, n \in B\}$ have all submeasure at least ε and they are decreasing with empty intersection. Since ϕ is a Maharam submeasure, this is impossible.

Or there is an infinite set D such that $[s_m]$ $(m \in D)$ form a decreasing chain. The intersection $\bigcap_{m \in D}[s_m]$ is a singleton, $\{x\}$, and again by the continuity of the submeasure, $\phi(\{x\}) \geq \varepsilon$. Thus $\{x\} \notin I$, contradiction. $\square$

Let $f(n) = m_{2^{-n}}$ as given by the previous Claim. Interpret the Cohen forcing as adding a function $g \in \prod_n 2^{f(n)}$ with finite conditions. Let $D_m = \bigcup_{n > m}[g(n)]$. It is not difficult to see that $V \cap 2^\omega \subset D_m$ for every number $m \in \omega$ and the submeasures $\phi(D_m)$ converge to zero. Therefore $\bigcap_m D_m$ is a submeasure zero set containing all the ground model reals.

Finally, to show that $I \perp J$ note that by a result of Shelah [20] the poset P_J adds a Cohen real. The argument is concluded in a manner similar to Lemma 3.4. $\square$

Proof of Theorem 3.3. Let I, J be definable c.c.c. σ-ideals, and suppose that the first two alternatives in the Theorem fail. Use the c.c.c. to find partitions $2^\omega = B_0 \cup B_1$ and $2^\omega = C_0 \cup C_1$ into Borel sets such that P_I below B_0 and P_J below C_0 are bounding forcings while the posets P_I below B_1 and P_J below C_1 add an unbounded real. Pick i, j such that $B_i \notin I$ and $C_j \notin J$. If $i = j$ we may assure that if P_I is `meager` (`null`, respectively) below B_i then P_J is not `meager` (`null`, respectively) below C_j. In either case, by one of lemmas 3.4, 3.5 or 3.6 we are in the situation of Lemma 3.2. $\square$

4   Concluding remarks

Another corollary of Theorem 0.1 precisely determines the extent of ccc-ness of a weakly distributive definable forcing P_I. Recall that a subset F of a poset $\mathbb{P}$ is n-linked if every n-element subset of F has a lower bound, and that $\mathbb{P}$ is *σ-n-linked* if it can be covered by countably many n-linked sets. An $F \subseteq \mathbb{P}$ is centered if every finite subset of F has a lower bound, and $\mathbb{P}$ is σ-centered if it can be covered by countably many centered subsets. It is well-known that all these chain conditions are different. Also, by a result of Todorcevic ([24], see also [2, 3.6.C]), there is a Borel ccc poset that is not σ-2-linked.

Corollary 4.1 (LC). *If I is a σ-ideal of Borel sets and P_I is weakly distributive, then the following hold.*

1. P_I is not σ-centered.

2. If I is moreover definable, then P_I is ccc if and only if it is σ-n-linked for all n.

Proof. Assume $\mathcal{B}$ is σ-centered and fix centered sets X_n maximal under the inclusion whose union covers $\mathcal{B}$. Since by Fact 1.1 every positive set has a compact subset the intersection of each X_n is a singleton. This implies that a co-countable set belongs to I, a contradiction.

Only clause 2 requires the definability and large cardinal assumptions. By Theorem 0.1 it suffices to prove a well-known fact that if ϕ is a Maharam submeasure on Borel algebra of 2^ω and $\mathrm{Null}(\phi)$ contains all countable sets, then the quotient algebra is σ-n-linked for all n (this is [4, Exercise 393Y(a)]). Recall first that it is completely generated by its countable subalgebra $\mathcal{B}_0$ given by the name for the P_I-generic real. Now consider the metric on $\mathcal{B}$ defined by $\rho(A,B) = \phi(A \Delta B)$. It is not difficult to check that $(\mathcal{B},\rho)$ is a complete metric space, and as an easy consequence of [4, 393B (c)], it is isomorphic to the completion of $(\mathcal{B}_0,\rho)$, and in particular separable. For $A \in \mathcal{B}_0$ the set

$$F_A = \{C | \rho(A,C) < \rho(0_{\mathcal{B}}, A)/n\}.$$

is n-linked, and $\bigcup_{A \in \mathcal{B}_0} F_A$ covers $\mathcal{B}$. $\square$

We conclude with a question asked by Solecki (personal communication).

Question 4.2. Are the following equivalent for every c.c.c. σ-ideal I on Borel subsets of 2^ω that is analytic on G_δ?

1. Compact sets are dense in P_I and I is ccc.

2. I is the null ideal of some Maharam submeasure.

If the answer is positive, this would strengthen Theorem 0.1 and nicely complement a result of [12] where it was proved that every ccc σ-ideal σ-generated by compact sets is Borel-isomorphic to `meager`.

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